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On Elliptic Modular Equations for Transformations of Orders 29, 31, 37.

By Arthur Berry.

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§ 1. *Introduction.*

The problem of the transformation of elliptic functions gives rise to the problem of calculating modular equations corresponding to a transformation of any order; *i. e.*, of algebraic equations connecting some modular function of the ratio of the periods (τ) with the same function of the ratio of the transformed periods, which can in general be taken to be $n\tau$, or τ/n, where n is the order of the transformation. Jacobi[*] used the modular function $u \equiv \sqrt[4]{k}$, for which Hermite introduced the notation $\phi(\tau)$, and computed the modular equations for $n = 3, 5$, as rational equations between u and the transformed modulus $v \equiv \sqrt[4]{\lambda}$. Sohncke[†] subsequently dealt with the cases of $n = 7, 11, 13, 17, 19$. As far as I know, no higher cases have been worked out in this form, but a large number of modular equations have been worked out in terms of various irrational functions of k, λ conjointly. The two most extensive sets of equations have been given by Schroeter,[‡] who used irrational functions differing from order to order and computed the equations for prime orders up to 31, as well as for certain non-prime orders, and by R. Russell,[§] who worked systematically with the functions $k\lambda$, $k'\lambda'$ and their square roots and fourth roots and obtained equations for prime orders up to 59, with the exception of 41 (for which case his work is not quite finished) and 37, as well as for several non-prime orders and for various higher prime orders. A number of irrational modular equations were also given a little earlier by E. W. Fiedler[||] in his inaugural dissertation.

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[*] *Fundamenta Nova*, §§ 13, 15.

[†] *Crelle's Journal*, vol. 16 (1836).

[‡] *De Aequationibus Modularibus*, Königsberg, 1854, and *Crelle's Journal*, vol. 58 (1860).

[§] *Proceedings of the London Mathematical Society*, Series I, vol. 19 (1889), vol. 21 (1891).

[||] Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen. Zürich, 1885 also in Wolf's *Zeitschrift*, vol. 30.

In 1858 Hermite[*] introduced a new modular function $\chi(\tau) \equiv \sqrt[12]{(kk')}$. The corresponding modular equations were given by Schläfli[†] for prime orders up to 19, and the equations frequently bear his name; the case of $n = 23$ was given by Weber[‡] as an illustration of the theory of complex multiplication.

Klein, in a well-known paper,[§] considered the modular equations formed with the absolute invariant J, and gave explicit formulae for J and the transformed function J′ as rational functions of a parameter, in the cases $n = 3, 4, 5, 7, 13$. In the same volume Gierster worked out the remaining cases in which the modular equation is of deficiency zero; viz., the cases of $n = 6, 8, 9, 10, 12, 16, 18, 25$.

A comparison of the modular equations formed respectively with J, u, χ shews that the first are functionally much the simplest and the last the most complicated. For example, the deficiency (genus) of the u, v equation for $n = 3$ is already 7, and for $n = 5$ is 15; while the deficiency of Schläfli's equation for $n = 5$ is 23. But the order of numerical simplicity is the reverse. The modular equation in J for $n = 3$, which I believe to be the highest which has been explicitly calculated,[||] consists of 17 terms and contains the numerical coefficient $2^{15}.\,5^{6}.\,22973$; whereas for the higher case of $n = 5$ the u, v equation consists of only 6 terms with no coefficient greater than 5, and the Schläfli equation (with a slight numerical modification) assumes the extremely simple form

$$x^6 + y^6 - xy + 4x^5y^5 = 0.$$

The object of this paper is to establish what I believe to be a new property of the Schläfli modular equations (§3), and to compute the equations for the cases $n = 29, 31, 37$, the last case being one for which, as far as I know, no modular equation in any form has been computed.

§ 2. *The Modular Function* $x(\tau)$.

I find it convenient for numerical purposes slightly to modify Hermite's function and to work instead with

$$2^{-1/6}.\sqrt[12]{(kk')} \equiv 2^{-1/6}\,\chi(\tau) \equiv q^{1/24} / \prod_{1}^{\infty} (1 + q^{2m-1}).$$

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[*] Sur la résolution de l'équation du quatrième degré, *Comptes Rendus*, vol. 46 (1858), reprinted with other papers in the pamphlet, Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré. Paris (1859).

[†] *Crelle's Journal*, vol. 72 (1870).

[‡] In his book *Elliptische Functionen und Algebraische Zahlen*, § 99.

[§] Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades, *Mathematische Annalen*, vol. 14 (1879).

[||] It is given in a slightly different notation by H. J. S. Smith, *Proceedings of the London Mathematical Society*, vol. 9 (1878), and *Collected Mathematical Papers*, vol. 2, p. 242.

This is the reciprocal of Weber's function $f(\omega)$; I denote it for convenience by $x(\tau)$, and denote the corresponding transformed function $x(n\tau)$ by y, so that the required modular equation appears as an equation in x and y.

It is known that $x(-{}^1/\tau) = x(\tau)$; $x(2+\tau) = \varepsilon x(\tau)$, where $\varepsilon = e^{2i\pi/24}$. The modular equation for a transformation of prime order n, greater than 3, is of order $n+1$ in either x or y and symmetrical in them. The $n+1$ values of y are $y_\infty \equiv x(n\tau)$, and $y_r \equiv x((48r+\tau)/n)$ $(r = 0, 1 \ldots . n-1)$. When x vanishes all the values of y vanish also, and the corresponding approximations are given by $y^n = x$, and $y = x^n$. Also the modular equation is unaffected if x is replaced by εx, y by $\varepsilon^n y$; since a term y^{n+1} must occur, it at once follows that if a term $x^\alpha y^\beta$ occurs $\alpha + n\beta \equiv n+1$, *mod.* 48. The high modulus of this congruence explains why the number of terms is small compared with the number in the corresponding u, v or J, J′ equations. Further by means of the quadratic transformation $\{\tau, (1+\tau)/(1-\tau)\}$ it can be shewn that the modular equation is unaffected if in it we write $2^{-1/2}x^{-1}$, $\left(\frac{2}{n}\right) 2^{-1/2} y^{-1}$ for x, y respectively,* where $\left(\frac{2}{n}\right)$ is Legendre's symbol and denotes $+1$ or -1 as n is of the form $8m \pm 1$ or $8m \pm 3$. We shall refer to this property briefly as *reciprocity.*

These properties enable us to write down the possible terms in the modular equation, and to assign coefficients to four of them; viz., we always have the four terms

$$x^{n+1} + y^{n+1} - xy - \left(\frac{2}{n}\right) 2^{(n-1)/2} x^n y^n;$$

moreover, a large number of the remaining coefficients are not independent, since the coefficient of $x^\alpha y^\beta$ is equal to that of $x^\beta y^\alpha$, while that of $x^{n+1-\alpha} y^{n+1-\beta}$ can be at once derived from either by reciprocity. Lastly, the coefficients are all integers, and save for the four "known" terms just written down, they are all multiples of n.

§ 3. *The Branch Places of the* (x, y) *Modular Equation.*

The approximations at the origin and at infinity given in the last paragraph show that y, regarded as a function of x, has at $x = 0$ and $x = \infty$ n branches which are cyclically permuted when x describes a circuit round the place, and one distinct branch; in other words, the Riemann surface has at each of the places $x = 0$, $x = \infty$ a winding-point of order n and an isolated sheet.

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* These properties and their proofs are all to be found in Weber's book.

The remaining branch places are given by $x^{24} = 2^{-6}$, or $\tau = i + 2p$, where p is an integer. To prove this and to investigate the nature of the branching let us first consider $x = 2^{-1/4}$, $\tau = i$.

The substitution T or $(\tau, -1/\tau)$ applied to τ leaves $x(\tau)$ unchanged, and a slight modification of a familiar process shows that this substitution interchanges the roots y_0, y_∞ of the modular equation and also the roots y_r, y_s, where

$$1 + 48^2 . r . s \equiv 0, \ mod. \ n.$$

Thus y_∞ becomes

$$x(nT\tau) = x(-n/\tau) = x(\tau/n) = y_0.$$

Further, y_r becomes

$$x\left(\frac{48r + T\tau}{n}\right) = x\left(\frac{-1 + 48r\tau}{n\tau}\right) = x\left(\frac{-1 - 48^2 . r . s + 48r(48s + \tau)}{n\tau}\right).$$

If now we choose s a positive integer less than n to satisfy the congruence $1 + 48^2 rs \equiv 0$, *mod.* n, say $1 + 48^2 rs = cn$, where c is an integer, we have

$$x\left(\frac{-c + 48r . \tau_s}{-48s + n\tau_s}\right),$$

where τ_s is written for $\tau(48s + \tau)/n$. This expression is of the form

$$x(c + d\tau_s/a + b\tau_s),$$

where a, b, c, d are integers such that $ad - bc = 1$; also b, c are odd, a, d are even, and $b - c = n - \frac{1 + 48^2 rs}{n} = \frac{n^2 - 1 - 48^2 rs}{n} \equiv 0$, *mod.* 48, since n is a prime number other than 2 or 3. Hermite's formulæ for linear transformation of the function $x(\tau)$, or $\chi(\tau)$,[*] show that our expression reduces to $x(\tau_s)$ or y_s. Now the congruence $1 + 48^2 rs \equiv 0$, *mod.* n, always admits of a solution for any value of r from 1 to $n - 1$, since n is prime to 48; also the congruence is symmetrical in r, s; hence in general the substitution T permutes the two roots y_r, y_s. The exceptional case is when r coincides with s, in which case the corresponding root is unaffected. The congruence $1 + 48^2 r^2 \equiv 0$, *mod.* n, evidently admits of solutions under the same conditions as the congruence $1 + r^2 \equiv 0$, and therefore, by a well-known result of the theory of numbers,[†] there are two solutions or none as n is of the form $4p + 1$ or $4p - 1$. Hence the substitution T applied to τ interchanges in pairs $n - 1$ or $n + 1$ of the roots of the modular equation, while

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[*] See for example Tannery and Molk's *Théorie des Fonctions Elliptiques*, vol. 2, Table XLVI.

[†] Mathews' *Theory of Numbers*, § 37.

there are two or no isolated roots. Since $Ti = i$ and $x(i) = 2^{-1/4}$, we thus see that:

When $x = 2^{-1/4}$, *the Riemann surface has* $(n-1)/2$ *or* $(n+1)/2$ *simple winding-points, at each of which two roots are interchanged, while* 2 *or* 0 *sheets are distinct, according as n is of the form* $4p+1$ *or* $4p-1$.

Further, it can be shown that the two isolated values of y, when they occur (i. e., in the case $n = 4p+1$), are always equal to the value of x multiplied by ± 1 or $\pm i$. This can be established by using the quadratic transformation $\left(\tau, \frac{1+\tau}{1-\tau}\right)$, of which $\tau = i$ is a fixed point. This substitution converts x into $2^{-1/2}x^{-1}$, $x(n\tau)$ into $\left(\frac{2}{n}\right) 2^{-1/2}/x(n\tau)$.[*] Now if $\tau' = (1+\tau)/(1-\tau)$,

$$x\{(\tau' + 48r)/n\} = x\left(\frac{-n}{\tau' + 48r}\right) = x\left\{\frac{n(-1+\tau)}{1 + 48r + (1-48r)\tau}\right\}.$$

Choose r so that $1 + 48^2 r^2 \equiv 0$, *mod.* n, or $1 + 48^2 r^2 = nn'$, where n' is an integer. Then our function becomes $x\left\{\frac{-(1+48r)+\tau_r}{n' + (1-48r)\tau_r}\right\}$, (where $\tau_r = (\tau + 48r)/n$), which is $x\left(\frac{1+U\tau_r}{1-U\tau_r}\right)$, where $U\tau \equiv \frac{c+d\tau}{a+b\tau}$ and $2a = -1 - 48r + n'$, $2b = 1 - 48r + n$, $2c = -1 - 48r + n'$, $2d = -1 + 48r + n$. Here a, b, c, d are integers satisfying the relation $ad - bc = 1$, and it is easy to verify that $(a+d)(abd - c) \equiv 24$ or 0, *mod.* 48, according as n is of the form $8p-3$ or $8p+1$. Hence $x\left(\frac{1+U\tau}{1-U\tau}\right) = 2^{-1/2}/x(U\tau_r) = \left(\frac{2}{n}\right) . 2^{-1/2}/x(\tau_r)$ by Hermite's formulae for the linear transformation of $x(\tau)$.

Now if $\tau = i$, $\tau' = i$ also, so that we have proved that for this value of τ, $y_r = \left(\frac{2}{n}\right) . 2^{-1/2}/y_r$, where r is either root of the congruence $1 + 48^2 r^2 \equiv 0$, *mod.* n. Thus, if $n = 8p-3$, $y_r = \pm i 2^{-1/4}$, and, if $n = 8p+1$, $y_r = \pm 2^{-1/4}$. Moreover, the two isolated roots of the equation, corresponding to the two roots of the congruence, are of the form $x(\pm\alpha + i\beta)$, so that by a known result one is always conjugate to the other. Hence in one case the two isolated roots are $i2^{-1/4}$ and $-i2^{-1/4}$ respectively, and in the other they are both $2^{-1/4}$ or both $-2^{-1/4}$.

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[*] Weber, *loc. cit.*, § 76.

I have not succeeded in finding any simple criterion distinguishing these last two cases. It is however quite easy in any concrete case to reduce $y_r \equiv x\,\{(i+48r)/n\}$ by continual application of the substitutions T or $(\tau, -1/\tau)$ and S^2 or $(\tau, 2+\tau)$ to the form $\varepsilon^{\lambda} x(i)$. For example, if $n = 113$ the roots of the congruence are $r = 35$ or 78; now using in succession the identities $98^2 + 1 = 113.85$, $72^2 + 1 = 85.61$, $50^2 + 1 = 61.41$, $32^2 + 1 = 41.25$, $18^2 + 1 = 25.13$, $8^2 + 1 = 13.5$, we obtain

$$x\left(\frac{i+48.35}{113}\right) = \varepsilon^7 x\left(\frac{i+98}{113}\right) = \varepsilon^7 x\left(\frac{85}{-i+98}\right) = \varepsilon^7 x\left(\frac{i-98}{85}\right)$$

$$= \varepsilon^6 x\left(\frac{i+72}{85}\right) = \varepsilon^6 x\left(\frac{61}{-i+72}\right) = \varepsilon^6 x\left(\frac{i-72}{61}\right) = \varepsilon^5 x\left(\frac{i+50}{61}\right)$$

$$= \varepsilon^5 x\left(\frac{41}{-i+50}\right) = \varepsilon^5 x\left(\frac{i-50}{41}\right) = \varepsilon^4 x\left(\frac{i+32}{41}\right) = \varepsilon^4 x\left(\frac{25}{-i+32}\right)$$

$$= \varepsilon^4 x\left(\frac{i-32}{25}\right) = \varepsilon^3 x\left(\frac{i+18}{25}\right) = \varepsilon^3 x\left(\frac{13}{-i+18}\right) = \varepsilon^3 x\left(\frac{i-18}{13}\right)$$

$$= \varepsilon^2 x\left(\frac{i+8}{13}\right) = \varepsilon^2 x\left(\frac{5}{-i+8}\right) = \varepsilon^2 x\left(\frac{i-8}{5}\right) = \varepsilon x\left(\frac{i+2}{5}\right)$$

$$= \varepsilon x\left(\frac{1}{-i+2}\right) = x(i) = 2^{-1/4}, \text{ so that in this case } \lambda = 24.$$

I have tested in this way all the primes of the forms $4p+1$ up to 113. In all those of the form $8p-3$, λ is ± 6 in accordance with the theory; for $n = 17, 41, 97$, λ is 12; and for $n = 73, 89, 113$, λ is 24.

It is obvious that exactly like properties hold at each of the 24 points given by $x^{24} = 2^{-6}$, and it is easily seen that there are no other branch places, since a branch place is clearly a fixed point of a substitution belonging to that subgroup of the modular group $\left(\tau, \dfrac{c+d\tau}{a+b\tau}\right)$ which leaves $x(\tau)$ unaltered. From the known properties of the modular group it readily follows that of the three fundamental singular points $\tau = \infty$, $\tau = i$, $\tau = \dfrac{-1+i\sqrt{3}}{2}$ of the modular group, the last gives rise to no fixed point of a substitution belonging to our subgroup, while the first two give the branch places 0, ∞, $\sqrt[24]{2^{-6}}$ already considered.

§ 4. *The Modular Equation for $n = 29$.*

Using the properties quoted in § 2, we have for the equation

$$
\begin{aligned}
&- xy\,(1 - 2^{14} x^{28} y^{28}) \\
&+ 29ax^5 y^5\,(1 - 2^{10} x^{20} y^{20}) \\
&+ 29b\,(x^{10} y^4 + x^4 y^{10})\,(1 + 2^8 x^{16} y^{16}) \\
&+ 29\,(c_1 x^{15} y^3 + c_2 x^9 y^9 + c_1 x^3 y^{15})\,(1 - 2^6 x^{12} y^{12}) \\
&+ 29\,(d_1 x^{20} y^2 + d_2 x^{14} y^8 + d_2 x^8 y^{14} + d_1 x^2 y^{20})\,(1 + 2^4 x^8 y^8) \\
&+ 29\,(e_1 x^{25} y + e_2 x^{19} y^7 + e_3 x^{13} y^{13} + e_2 x^7 y^{19} + e_1 xy^{25})\,(1 - 2^2 x^4 y^4) \\
&+ x^{30} + y^{30} + 29\,(f_1 x^{24} y^6 + f_2 x^{18} y^{12} + f_2 x^{12} y^{18} + f_1 x^6 y^{24}) = 0,
\end{aligned}
$$

where $a, b, c_1, c_2, d_1, d_2, e_1, e_2, e_3, f_1, f_2$ are eleven integers which have to be determined.

Substituting for x, y the q functions, $q^{1/24}/(1 + q)\ (1 + q^3) \ldots.$ and $q^{29/24}/\,(1 + q^{29})\ (1 + q^{3.29}) \ldots.$, we have an identity in q. It is convenient to divide by xy so as to get rid of fractional powers, and then to multiply by a power of $(1 + q)\,(1 + q^3) \ldots.$, so as to avoid as far as possible high indices in the binomials. If we retain only terms up to q^5 we have 5 equations giving in succession $e_1 = 1$, $d_1 = 9$, $c_1 = 27$, $b = 23$, $a = 7$.

We next consider the factors of the modular equation corresponding to the case of complex multiplication given by $y = x$. Making this substitution and writing for brevity z for x^4, the modular equation reduces to

$$
\begin{aligned}
f(z) \equiv -1 + 29az^2 + 58bz^3 + 29\,Cz^4 + 29Dz^5 + 29Ez^6 + (2 + 29F)z^7 \\
- 2^2 .\, 29 \,.\, E \,.\, z^8 + \ldots. + 2^{14} z^{14} = 0,
\end{aligned}
$$

where $C = 2c_1 + c_2$, $D = 2\,(d_1 + d_2)$, $E = 2e_1 + 2e_2 + e_3$, $F = 2\ (f_1 + f_2)$, and the coefficients not written down can be at once supplied if wanted from reciprocity.

1. Corresponding to the expression of 29 as the quadratic form $4^2 + 13$, we have

$$
x\left(\frac{4 + i\sqrt{13}}{29}\right) = x\left(\frac{1}{4 - i\sqrt{13}}\right) = x(-4 + i\sqrt{13}) = \varepsilon^{-4} x(4 + i\sqrt{13}),
$$

and it readily follows that

$$
x\left(\frac{48 + (6 + i\sqrt{13})}{29}\right) = \varepsilon x\left(\frac{-4 + i\sqrt{13}}{29}\right) = \varepsilon x\,(4 + i\sqrt{13}) = x\,(6 + i\sqrt{13}.)
$$

Thus $y_1 \equiv x\left(\dfrac{48 + \tau}{29}\right) = x\,(\tau)$, when $\tau = 6 + i\sqrt{13}$, and similarly $y_6 = x\,(\tau)$.

Thus $z = x^4(6 + i\sqrt{13})$ is a repeated root of $f(z) = 0$. But the modular equation for $n = 13$ gives $x^4(i\sqrt{13}) = \frac{\sqrt{13} - 3}{4}$, so that $z = \frac{-\sqrt{13} + 3}{4}$; rationalizing this we have $4z^2 - 6z - 1$ as a repeated factor of $f(z)$.

2. Similarly from the expression of 29 as $1^2 + 28$, we deduce

$$x\left(\frac{28 + i\sqrt{28}}{2\,.\,29}\right) = x\left(\frac{-14 + i\sqrt{7}}{7}\right) = \varepsilon^{-1}x(i\sqrt{7}),$$

whence

$$y_2 = x\left(\frac{2\,.\,48 + (6 + i\sqrt{7})}{29}\right) = \varepsilon^2 x\left(\frac{-14 + i\sqrt{7}}{29}\right) = \varepsilon^3 x(i\sqrt{7}) = x(6 + i\sqrt{7}),$$

and similarly $y_5 = x$ for the same value of the argument. From the modular equation for $n = 7$ we have $x(i\sqrt{7}) = 1/\sqrt{2}$, so that $z \equiv x^4(6 + i\sqrt{7}) = -1/4$; we have therefore the repeated factor $4z - 1$, and by reciprocity we have associated with this the factor $z + 1$, and therefore the repeated factor $4z^2 + 3z - 1$.

3. We have obviously $x\left(\frac{i\sqrt{29}}{29}\right) = x(i\sqrt{29})$, *i. e.* $y_0 = x$, when $\tau = i\sqrt{29}$; and there are also associated with this two values $\frac{\pm 4 + i\sqrt{29}}{3}$, for which we have respectively $y_4 = x$ and $y_{25} = x$. We might quote the values of $x(i\sqrt{29})$ from Weber's table,* but I purposely use only results connected with modular equations of lower order than the one under discussion.

We have now accounted for all the factors of $f(z)$, so that

$$-f(z) \equiv (1 + 6z - 4z^2)^2\,(1 + 3z - 4z^2)^2\,(1 + \alpha z + \beta z^2 + \gamma z^3 - 2^2\beta z^4 + 2^4\alpha z^5 - 2^6 z^6),$$

where the last factor corresponds to the values just discussed.

Equating coefficients of z, z^2, z^3, we have

$$\begin{aligned} 0 &= 18 + \alpha \\ -29a &= 101 + 18\alpha + \beta \\ -58b &= 108 + 101\alpha + 18\beta + \gamma, \end{aligned}$$

whence, a and b being known, $\alpha = -18$, $\beta = 20$, $\gamma = 16$. We have thus shown that $x(i\sqrt{29})$ satisfies the sextic equation

$$1 - 18z + 20z^2 + 16z^3 - 2^2.\,20z^4 - 2^4.\,18z^5 - 2^6 z^6 = 0,$$

which agrees with Weber's equation.

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* *Loc. cit.*, pp. 499–504.

Equating coefficients of z^4, z^5, z^6, z^7, we now deduce

$$C = 8, \quad D = -376, \quad E = 432, \quad F = 2{,}966,$$

whence

$$c_2 = -46, \quad d_2 = -197, \quad 2e_2 + e_3 = 430, \quad f_1 + f_2 = 1{,}483.$$

We now want one more equation connecting e_2 and e_3 and one connecting f_1 and f_2. These might be obtained from the complex theory arising from $y = \varepsilon x$ or $y = \varepsilon^2 x$, and I originally obtained the missing coefficients by this method; but it is perhaps a little simpler to use the property given by § 3.

When $\tau = i$, or $x = 2^{-1/4}$, two of the roots of the modular equation are isolated and equal to $\pm i2^{-1/4}$; the rest are equal in pairs. On substituting $x = 2^{-1/4}$ and, for convenience, $y = 2^{-1/4}\eta$, we have consequently

$$\begin{aligned}
1 &+ \eta\,(2\,.\,29\,.\,e_1 - 2^7) + \eta^2\, 2^2.\,29\,.\,d_1 + \eta^3.\,2^3.\,29\,.\,c_1 + \eta^4.\,2^4.\,29\,.\,b \\
&+ \eta^5.\,29\,(2^5 a - 2e_1) + \eta^6.\,29\,.\,f_1 + \eta^7.\,2\,.\,29\,.\,e_2 + \eta^8.\,2^2.\,29\,.\,d_2 \\
&+ \eta^9.\,2^3.\,29\,.\,c_2 + \eta^{10}.\,29\,.\,(2^2 d_1 + 2^4 b) - \eta^{11}\,2\,.\,29\,.\,e_2 + \eta^{12}.\,29\,.\,f_2 \\
&+ \eta^{13}.\,2\,.\,29\,.\,e_3 + \eta^{14}.\,2^2.\,29\,.\,d_2 + \eta^{15}.\,0 + \eta^{16}.\,2^2.\,29\; d_2 - \eta^{17}.\,2\,.\,29\,.\,e_3 \ldots\ldots \\
&+ \eta^{29}\,(-2\,.\,29\,.\,e_1 + 2^7) + \eta^{30} = (1 + \eta^2)\,(1 + k_1\eta + k_2\eta^2 + k_3\eta^3 \ldots\ldots \\
&+ k_7\eta^7 - k_6\eta^8 + k_5\eta^9 - k_4\eta^{10} + k_3\eta^{11} - k_2\eta^{12} + k_1\eta^{13} - \eta^{14})^2.
\end{aligned}$$

Equating in this identity the coefficients of $\eta, \eta^2, \ldots\ldots, \eta^5$, we obtain successively $k_1 = -35$, $k_2 = -91$, $k_3 = -18$, $k_4 = 44$, $k_5 = -46$. Equating coefficients of η^8, η^9, we obtain $k_6 = 100$ and $k_7 = +54$. Equating coefficients of η^6, η^7, we then obtain $f_1 = 185$, $e_2 = 0$, whence, from the values already found, $e_3 = 430$, $f_2 = 1298 = 2\,.\,11\,.\,59$. Thus all the coefficients of the modular equation have been obtained. If in our identity we further equate coefficients of η^{10}, η^{11}, η^{12}, η^{13}, η^{14}, we obtain 5 new relations between the coefficients which may serve as equations of verification.

Collecting the results, we see that the modular equation for $n = 29$ is of the form written at the beginning of this paragraph, with the numerical coefficients:

$$\begin{aligned}
a = 7, \quad b = 23, \quad c_1 = 27, \quad c_2 = -46, \quad d_1 = 9, \quad d_2 = -197,& \\
e_1 = 1, \quad e_2 = 0, \quad e_3 = 430, \quad f_1 = 185, \quad f_2 = 1{,}298 = 2.11\,.\,59.&
\end{aligned}$$

§ 5. *The Modular Equation for* $n = 31$.

As before, the equation is

$$
\begin{aligned}
&-xy\,(1 + 2^{15}x^{30}y^{30})\\
&+ 31\,.\,a\,.\,x^4y^4\,(1 + 2^{12}x^{24}y^{24})\\
&+ 31\,(b_1x^{11}y^3 + b_2x^7y^7 + b_1x^3y^{11})\,(1 + 2^9x^{18}y^{18})\\
&+ 31\,(c_1x^{18}y^2 + c_2x^{14}y^6 + c_3x^{10}y^{10} + c_2x^6y^{14} + c_1x^2y^{18})\,(1 + 2^6x^{12}y^{12})\\
&+ 31\,(d_1x^{25}y + d_2x^{21}y^5 + d_3x^{17}y^9 + d_4x^{13}y^{13} + d_3x^9y^{17} + d_2x^5y^{21}\\
&\qquad\qquad + d_1xy^{25})\,(1 + 2^3x^6y^6)\\
&+ x^{32} + y^{32} + 31\,(e_1x^{28}y^4 + e_2x^{24}y^8 + e_3x^{20}y^{12} + e_4x^{16}y^{16} + e_3x^{12}y^{20}\\
&\qquad\qquad + e_2x^8y^{24} + e_1x^4y^{28}) = 0,
\end{aligned}
$$

where $a, b_1, b_2, c_1, c_2, c_3, d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4$ are fourteen integers to be determined.

If we substitute, as before, the q-products for x, y, divide out by xy, multiply by $\{(1 + q)(1 + q^3)\ldots.\}^{20}$, and expand as far as q^7, we find successively

$$d_1 = 1,\quad c_1 = 8,\quad b_1 = 13,\quad a = 3,\quad e_1 = 25,\quad d_2 = -256,\quad c_2 = -205.$$

Let us consider next the complex multiplications of the type $y = \varepsilon^2x$, which arise from the resolutions $31 = 2^2 + 27$, $31 = 4^2 + 15$. We verify at once that $x\{(-1 + i\sqrt{27})/31\} = \varepsilon^2x(-2 + i\sqrt{27})$, so that $\varepsilon^{-1}x(i\sqrt{27})$ satisfies the equation $y = \varepsilon^2x$. From the known modular equation for $n = 3$ we find that $z = x^6(i\sqrt{27})$ satisfies the cubic equation $1 - 60z + 48z^2 - 64z^3 = 0$.

Similarly [illegible] $(9 . 48 + 6 + i\sqrt{15})/31\} = \varepsilon^2x(6 + i\sqrt{15})$, so that $x(6 + i\sqrt{15}) = \varepsilon^3x(i\sqrt{15})$ is also a solution of $y = \varepsilon^2x$. From the modular equation of order 19 we have $x^6(i\sqrt{19}) = \dfrac{3 - \sqrt{5}}{24}$, so that $z = x^6(i\sqrt{19})$ satisfies the quadratic equation $1 - 24z + 64z^2 = 0$.

If in the modular equation we put (to avoid imaginaries) $x = \varepsilon\xi$, $y = \varepsilon^3\xi$, we get an equation of degree 10 in $z \equiv \xi^6$, satisfied by $x^6(i\sqrt{19})$ and $x^6(i\sqrt{27})$, of which we have just found a quadratic and a cubic factor. Using reciprocity we obtain the remaining factors.

We thus have the identity

$$
\begin{aligned}
&1 - 31az - 31B'z^2 - 31C'z^3 - 31D'z^4 + (1 - 31E')z^5 \ldots. + 2^{15}z^{10}\\
&\quad \equiv (1 - 60z + 48z^2 - 64z^3)\,(1 - 6z + 60z^2 - 8z^3)\,(1 - 24z + 64z^2)\,(1 - 3z + z^2),
\end{aligned}
$$

where

$$
\begin{aligned}
B' &= -b_1 + b_2, & C' &= -c_1 - c_2 + c_3,\\
D' &= 2d_1 - d_2 - d_3 + d_4, & E' &= 2e_1 - e_2 - e_3 + e_4.
\end{aligned}
$$

Equating coefficients, we obtain

$a = 3$ (verifying the previous result),

$$B' = -77, \quad C' = 834, \quad D' = -6{,}100, \quad E' = 24{,}295;$$

whence

$$b_2 = -64, \quad c_3 = 637, \quad d_3 - d_4 = 6{,}358, \quad e_2 + e_3 - e_4 = -24{,}245.$$

We can now obtain two more equations from the complex multiplication of the type $y = x$. It is easily verified that if

$$\tau = \frac{1 + i\sqrt{3}}{2}, \text{ each of } y_{27} \equiv x\left(\frac{27 \cdot 48 + \tau}{31}\right) \text{ and } y_7 \equiv x\left(\frac{7 \cdot 48 + \tau}{31}\right)$$

is equal to $x(\tau)$; so that $x\left(\frac{1 + i\sqrt{3}}{2}\right)$ is a repeated root of $y = x$, and this quantity is well known to be $2^{-1/6}$. We have also obviously

$$x\left(\frac{i\sqrt{31}}{31}\right) = x(i\sqrt{31}),$$

so that $x(i\sqrt{31})$ is another root of our equation, but we do not assume this quantity to be known.

Using reciprocity, we now have, on putting $y = x$ in the modular equation, and then writing z for x^6,

$$(1 - 31az - 31\ Bz^2 - 31\ Cz^3 - 31\ Dz^4 - (2 + 31E)\,z^5 \ldots + 2^{15}\,z^{10})$$
$$\equiv (1 - 2z)^2\,(1 - 4z)^2\,(1 + \alpha z + \beta z^2 + \gamma z^3 + 2^3\,\beta z^4 + 2^6\,\alpha z^5 + 2^9\,z^6),$$

where α, β, γ are at present unknown, and

$$B = 2b_1 + b_2 = -38, \qquad C = 2c_1 + 2c_2 + c_3 = 243,$$
$$D = 2d_1 + 2d_2 + 2d_3 + d_4, \qquad E = 2e_1 + 2e_2 + 2e_3 + e_4.$$

Equating coefficients of z, z^2, z^3, we have

$$\alpha - 12 = -31 \cdot a, \qquad \beta - 12\alpha + 52 = 31 \cdot 38,$$
$$\gamma - 12\beta + 52\alpha - 96 = -31 \cdot 241;$$

whence

$$\alpha = -81, \qquad \beta = 154, \qquad \gamma = -1{,}377.$$

Equating coefficients of z^4, z^5, we deduce

$$D = -1{,}084, \qquad E = 3{,}598;$$

whence

$$d_3 = 1{,}928 = 2^3 \cdot 241, \; d_4 = -4{,}430, \; e_4 = 17{,}346 = 2 \cdot 3 \cdot 7^2 \cdot 59, \; e_2 + e_3 = -6{,}899.$$

We have shown incidentally that $z = x^6(i\sqrt{31})$ satisfies the equation

$$1 - 81z + 154z^2 - 1{,}377z^3 + 2^3.\,154z^4 - 2^6.\,81z^5 + 2^9z^6 = 0,$$

agreeing with Weber's result.

We still want one more equation connecting e_2 and e_3. This can be obtained from complex multiplication of the type $y = \varepsilon x$. Corresponding to the resolution $31 = 5^2 + 6$, we easily find that if

$$\tau = -1 + i\sqrt{6}, \quad y_4 \equiv x\left(\frac{\tau + 4\,.\,48}{31}\right) = \varepsilon x(\tau),$$

so that $x(-1 + i\sqrt{6})$ is a solution of $y = \varepsilon x$. But from the modular equation for $n = 7$ we have $x^6(-1+i\sqrt{6}) = \varepsilon^{-3}(2 - \sqrt{2})/4$. If therefore we put $y = \varepsilon x$ and then $x^6 = \varepsilon^{-3}z$, the resulting equation in z is satisfied by $(2 - \sqrt{2})/4$. The equation is

$$-1 - 2^{15}z^{10} + 31a\,z(1 + 2^{12}z^8) + 31B''z^2(1 + 2^9z^6) + 31C''z^3(1 + 2^6z^4) \\ + 31D''z^4(1 + 2^3z^2) + (-1 + 31E'')z^5 = 0,$$

where

$$B'' = b_1 + b_2, \qquad C'' = -c_1 + c_2 + c_3,$$
$$D'' = -2d_1 - d_2 + d_3 + d_4, \qquad E'' = -2e_1 - e_2 + e_3 + e_4.$$

Putting $z = (2 - \sqrt{2})/4$ and substituting the known values of a, B'', C'', D'', we obtain

$$E'' = 7{,}903, \quad \text{whence} \quad e_2 - e_3 = 9{,}393,$$

and, since we know $e_2 + e_3$,

$$e_2 = 1{,}247, \qquad e_3 = -8{,}146.$$

Summing up the results, we see that the modular equation for $n = 31$ is of the form written at the beginning of this paragraph, with the numerical coefficients:

$$a = 3, \quad b_1 = 13, \quad b_2 = -64, \quad c_1 = 8, \quad c_2 = -205, \quad c_3 = 637 = 7^2.\,13,$$
$$d_1 = 1, \quad d_2 = -256, \quad d_3 = 1{,}928 = 2^3.\,241, \quad d_4 = -4{,}430, \quad e_1 = 25,$$
$$e_2 = 1{,}247 = 29\,.\,43, \quad e_3 = -8{,}146, \quad e_4 = 17{,}346 = 2\,.\,3\,.\,7^2.\,59.$$

I have checked the accuracy of these results by carrying the expansions as far as q^{13}, and further by verifying that when $\tau = i$, $x = 2^{-1/4}$, the roots of the modular equation are all equal in pairs.

§ 6. *The Modular Equation for $n = 37$.*

The equation is

$$
\begin{aligned}
&-xy\,(1-2^{18}x^{36}y^{36})\\
&+37\,(a_1x^{12}y^2+a_2x^{10}y^4+a_3x^8y^6+a_3x^6y^8+a_2x^4y^{10}+a_1x^2y^{12})\,(1+2^{12}x^{24}y^{24})\\
&+37\,(b_1x^{25}y+b_2x^{23}y^3+b_3x^{21}y^5+b_4x^{19}y^7+b_5x^{17}y^9+b_6x^{15}y^{11}+b_7x^{13}y^{13}\\
&\qquad+b_6x^{11}y^{15}+b_5x^9y^{17}+b_4x^7y^{19}+b_3x^5y^{21}+b_2x^3y^{23}+b_1xy^{25})\,(1-2^6x^{12}y^{12})\\
&+x^{38}+y^{38}+37\,(c_1x^{36}y^2+c_2x^{34}y^4+\ldots\ldots+c_9x^{20}y^{18}+c_9x^{18}y^{20}+\ldots\ldots\\
&\qquad\qquad\qquad+c_2x^4y^{34}+c_1x^2y^{36})=0,
\end{aligned}
$$

where $a_1, a_2, a_3, b_1, \ldots\ldots, b_7, c_1, \ldots\ldots, c_9$ are nineteen integers to be determined.

If we substitute, as before, q-products for x, y, divide by xy, multiply by $\{(1+q)(1+q^3)\ldots\ldots\}^{22}$, and expand as far as q^{12}, we get in succession

$$
\begin{array}{llll}
b_1=1, & a_1=5, & c_1=3, & b_2=-108,\\
a_2=51, & c_2=119, & b_3=-333, & a_3=133,\\
c_3=2{,}073, & b_4=540, & c_4=16{,}558, & b_5=4{,}806,
\end{array}
$$

and have also one superfluous equation serving as a verification.

It is now possible to complete the calculation by means of the property of § 3. When $\tau = i$, $x = 2^{-1/4}$, we have two isolated roots, viz. $y = \pm i2^{-1/4}$, of the modular equation, and the other roots are equal in pairs. If therefore we put $x = 2^{-1/4}$, $y = 2^{-1/4}\eta$, the left-hand side of the modular equation reduces to $1 + \eta^2$ multiplied by the square of a polynomial in η of degree 18. By reciprocity only 9 coefficients in this polynomial are independent, and by means of the known coefficients $a_1, a_2, a_3, b_1, \ldots\ldots, b_5, c_1, \ldots\ldots, c_4$, they can readily be computed by equating coefficients, or by extracting a square root. The left-hand side of the modular equation is thus found to reduce to

$$
\begin{aligned}
&(1+\eta^2)\,(1-108\eta+143\eta^2-432\eta^2-270\eta^4-792\eta^5-1{,}026\eta^6\\
&\qquad-864\eta^7-605\eta^8-900\eta^9+605\eta^{10}-864\eta^{11}\ldots\ldots-\eta^{18})^2.
\end{aligned}
$$

Equating coefficients of η^{10}, η^{11}, η^{12}, η^{13}, η^{14}, η^{16}, η^{18}, we obtain in turn

$$
\begin{array}{llll}
c_5=66{,}994, & b_6=11{,}556, & c_6=157{,}454, & b_7=15{,}031,\\
c_7=221{,}234, & c_8=179{,}655, & c_9=88{,}617. &
\end{array}
$$

By equating the coefficients of η^{15}, η^{17} we have two equations of verification.

Thus, finally, the modular equation is as written at the beginning of this paragraph, with the numerical coefficients:

$a_1 = 5$, $a_2 = 51$, $a_3 = 133 = 7\,.\,19$; $b_1 = 1$, $b_2 = -108$, $b_3 = -333 = -3^2.\,37$, $b_4 = 540$, $b_5 = 4{,}806 = 2\,.\,3^3.\,89$, $b_6 = 11{,}556 = 2^2.\,3^3.\,107$,

$b_7 = 15{,}031$, $c_1 = 3$, $c_2 = 119 = 7 \,.\, 17$, $c_3 = 2{,}073 = 3 \,.\, 691$, $c_4 = 16{,}558 = 2 \,.\, 17 \,.\, 487$, $c_5 = 66{,}994 = 2 \,.\, 19 \,.\, 41 \,.\, 43$, $c_6 = 157{,}454 = 2 \,.\, 11 \,.\, 17 \,.\, 421$, $c_7 = 221{,}234 = 2 \,.\, 13 \,.\, 67 \,.\, 127$, $c_8 = 179{,}655 = 3 \,.\, 5 \,.\, 7 \,.\, 29 \,.\, 59$, $c_9 = 88{,}617 = 3 \,.\, 109 \,.\, 271$.

We can now use complex multiplication for further verification of the coefficients. I have in fact worked out most of the cases, but I reproduce only one. Corresponding to the resolution $37 = 3^2 + 2^2 \,.\, 7$, it is easily verified that, if $\tau = 2 + i\sqrt{7}$, $y_5 = y_{35} = x(\tau)$; so that $x(2 + i\sqrt{7}) \equiv \epsilon x(i\sqrt{7})$ is a repeated root of the equation $y = x$. Making this substitution in the modular equation and writing for brevity $z = x^{12}$, we have

$$f(z) \equiv 1 - 2^{18} z^6 - 37Az(1 + 2^{12} z^4) - 37Bz^2(1 - 2^6 z^2) - (2 + 37C)z^3 = 0,$$

where

$$A = 2(a_1 + a_2 + a_3), \quad B = 2(b_1 + \dots + b_6) + b_7, \quad C = 2(c_1 + \dots + c_9).$$

From the modular equation for $n = 7$ we have $x(i\sqrt{7}) = 2^{-1/2}$, whence $z = -2^{-6}$; associated with this by reciprocity we have $z = 1$.

Thus $f(z) \equiv (1 + 63z - 2^6 z^2)^2 (1 + \lambda z - 2^6 z^2)$, where the second factor corresponds to $z = x^{12}(i\sqrt{37})$.

Giving A its known value 378 and equating coefficients of z, we have $\lambda = -14{,}112$. Equating coefficients of z^2, z^3, we have $B = 47{,}955$, $C = 1{,}465{,}414$, agreeing with preceding results. The residual factor gives

$$8z = -882 + 145\sqrt{37} = (\sqrt{37} - 6)^3;$$

so that

$$2x^4(i\sqrt{37}) = \sqrt{37} - 6,$$

which agrees with Weber's result.

I have to express my thanks to Miss H. P. Hudson, of Newnham College, who has helped me materially by carrying out some of the calculations independently.

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